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(a) $(\mathbb{Q}, d)$ is a metric space where $d(x, y) = |x - y|/(1 + |x - y|)$. Find (with justification) the completion of $\mathbb{Q}$ with respect to this metric.

Solution. We claim that the completion of this metric is $(\mathbb{R}, d)$. We make this claim with the knowledge that the completion of $(\mathbb{Q}, |\cdot|)$ is $(\mathbb{R}, |\cdot|)$ and the fact that given an $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ s.t.

$$d(x, y) < \epsilon \iff |x - y| < \eta(\epsilon)$$

for $x, y \in \mathbb{Q}$.

To complete the argument, we need 2 things: (i) an isometry $\phi : (\mathbb{Q}, d) \to (\mathbb{R}, d)$ and (ii) $\phi(\mathbb{Q})$ must be dense in $\mathbb{R}$. For (i), pick $\phi(x) = x$. Clearly, this mapping is an isometry since the map is distance preserving and $\phi(\mathbb{Q})$ is dense in $\mathbb{R}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$ under the standard absolute value metric.

(b) $\phi : [0, 1] \to \mathbb{R}$ is a smooth non-negative function satisfying $\int_0^1 \phi(x)dx = 1$. Show that there is a unique continuous function which solves the functional equation

$$\mu(x) = \frac{2}{3}\int_0^x \phi(y)\mu(1 - y^2)dy + \phi(1 - x^3) \quad \forall x \in [0, 1]$$

Solution. Before using the Contraction Mapping Theorem, we need to make sure that we are working in a complete metric space. Take $(M, d) = (C([0, 1]), ||\cdot||_\infty)$. Take

$$T[\mu] = \frac{2}{3}\int_0^x \phi(y)\mu(1 - y^2)dy + \phi(1 - x^3)$$

Clearly, such T is continuous, since it is differentiable. We now need to show that this mapping is a contraction. This should be clear, since

$$\begin{aligned}
||T[\mu] - T[\nu]||_\infty &= \left|\left|\frac{2}{3}\int_0^x \phi(y)\mu(1 - y^2)dy + \phi(1 - x^3)\right.\right. \\
&- \left.\left.\left(\frac{2}{3}\int_0^x \phi(y)\nu(1 - y^2)dy + \phi(1 - x^3)\right)\right|\right|_\infty \\
&\le \frac{2}{3}\int_0^x \phi(y)|\mu(1 - y^2) - \nu(1 - y^2)|dx
\end{aligned}$$

Since we have a contraction, there exists a unique μ s.t. $T[\mu] = \mu$. Continuity of μ follows immediately.

Therefore, there is a unique continuous function which solves the given functional equation.

(c) f_n is a sequence of non-negative measurable functions, and $g(x) = \lim_{M\to\infty} \sum_{n=1}^{M} f_n(x)$. Show that

$$\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{x : f_m(x) \geq 1/k\} \subseteq \{x : g(x) = +\infty\}$$

Solution. Pick $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{x : f_m(x) \geq 1/k\}$. It should be clear that such an x satisfies the statement: "for any k, for all n, there exists an $m > n$ s.t. $f_m(x) \geq 1/k$".

Consider the sequence of partial sums defined by

$$g_m(x) = \sum_{n=1}^{m} f_n(x).$$

We can use the fact that f_n is non-negative and the fact that we can always find an index for any $k \in \mathbb{N}$ s.t. $f_m(x) \geq 1/k$ to conclude that the sequence of partial sums diverge to $+\infty$. Therefore, $x \in \{x : g(x) = +\infty\}$.

Therefore, $\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{x : f_m(x) \geq 1/k\} \subseteq \{x : g(x) = +\infty\}$

f and g are two uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Prove or give a counter-example: $f \circ g$ is also uniformly continuous.

Proof. This statement is true. Assume f, g are uniformly continuous. Given some $\epsilon > 0$ there is a $\delta_1 > 0$ s.t. $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta_1$. Also, there exists $\delta_2 > 0$ s.t. $|g(s) - g(t)| < \delta_1$ whenever $|s - t| < \delta_2$.

Let $\delta = \min(\delta_1, \delta_2) > 0$ and $s = f(x), t = f(y)$ from above. So, given an $\epsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta \implies |g(f(x)) - g(f(y))| < \epsilon$.

Therefore $f \circ g$ is uniformly continuous.

(e) Show that the following limit exists and compute it:

$$\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1/n} \left(1 - \frac{k}{n}\right)^n.$$

Solution. We would like to be able to move the limit inside the sum to ease calculation. That is, we would like to show

$$\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1/n} \left(1 - \frac{k}{n}\right)^n = \sum_{k=1}^{\infty} \lim_{n\to\infty} k^{-1/n} \left(1 - \frac{k}{n}\right)^n$$

To do this, we need a dominating function so that we can use dominated convergence. First, note that the $k^{-1/n} \leq 1$ and $(1 - k/n)^n \leq e^{-k}$ for all $k, n \in \mathbb{N}$. So, e^{-k} is a dominating function. Therefore,

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{n} k^{-1/n} \left(1 - \frac{k}{n}\right)^n &= \sum_{k=1}^{\infty} \lim_{n\to\infty} k^{-1/n} \left(1 - \frac{k}{n}\right)^n \\ &= \sum_{k=1}^{\infty} e^{-k} \end{aligned}$$

which is a geometric series converging to $\frac{1}{1-e^{-1}} = \frac{e}{e-1}$.